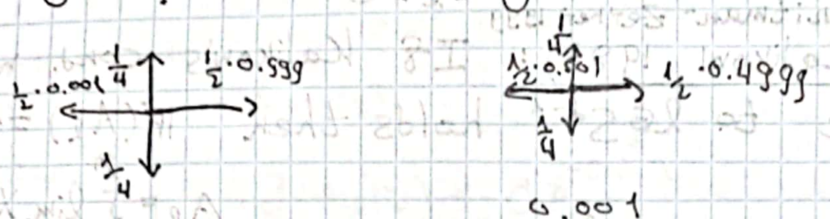
Lecture 7

Random walks in random env. in dimension $d \geq 2$:

<u>Kalikow example:</u> The transition prob. are IID, taking one of the following two possibilities



Prob. 0.999 $\qquad\qquad$ 0.001

How to prove that: $\mathbb{P}^0(\lim_{n\to\infty} X_n \cdot (1,0) = \infty) = 1$?

Kalikow's Condition

<u>Idea:</u> For each connected $U \subsetneq \mathbb{Z}^d$ with $0 \in U$, define a random walk in a <u>deterministic environ.</u> and show that if this walk has a drift, uniformly in U, then the original RWRE has a drift.

<u>Deterministic environ. walk:</u>

Let $U \subsetneq \mathbb{Z}^d$ connected, $0 \in U$.

$$\hat{\mathbb{P}}_U(x, x+e) = \frac{\mathbb{E}^0\left(\sum_{n=0}^{\tau_{U^c}} \mathbb{1}_{X_n = x}\, \omega(x, x+e)\right)}{\mathbb{E}^0\left(\sum_{n=0}^{\tau_{U^c}} \mathbb{1}_{X_n = x}\right)}$$

$x \in U$, $e \in \{\pm e_i\}_{i=1}^d$

$\tau_{U^c} = \min\{n \geq 0 : X_n \notin U\}$

Why is this walk useful?

We work with P IID and uniformly elliptic.

<u>Lemma:</u> If $\hat{P}_U(\tau_{U^c} < \infty) = 1$ then $\forall v \in U^c$:

$$\hat{\mathbb{P}}_U(X_{\tau_{U^c}} = v) = \mathbb{P}^0(X_{\tau_{U^c}} = v)$$

In particular, $\mathbb{P}^0(\tau_{U^c} < \infty) = 1$.

Proved this last time.

<u>In words:</u> The exit position from U has the same distribution under the deterministic environment and under the annealed RW measure.

<u>Kalikow's condition</u>: For $\ell \in S^{d-1}$

$$\varepsilon_\ell = \inf_{U, x \in U} \underbrace{\sum_{e \in \{\pm e_i\}} e \cdot \ell \; \hat{P}_U(x, x+e)}_{\text{Ex. drift in direction } \ell \text{ when starting in } x \text{ in } \hat{P}_U}$$

$U \subsetneq \mathbb{Z}^d$, U conn. $0 \in U$

The cond. is that $\varepsilon_\ell > 0$

Sznitman-Zerner 1999

<u>Thm</u> (Kalikow 1981): If Kalikow's cond. with respect to $\ell \in S^{d-1}$ holds then $\mathbb{P}^0(A_\ell) = 1$

$A_\ell = \{\lim_{n\to\infty} X_n \cdot \ell = \infty\}$

and moreover $\frac{X_n \cdot \ell}{n} \to v_\ell > 0$, $\mathbb{P}^0$-a.s

<u>proof of $\mathbb{P}^0(A_\ell) = 1$</u>:

<u>Idea</u>: Use Hoeffding's (Azuma's) ineq. to say that with very high prob. the walk follows the drift.

<u>Hoeffding's ineq.</u> Let $(X_k)_{k=0}^n$ be a super-martingale. Assume that the differences $|X_k - X_{k-1}| \le c_k$ for $1 \le k \le n$, almost surely. Then:

$$\mathbb{P}(X_n - X_0 \ge t) \le e^{-\frac{t^2}{2\sum_{k=1}^n c_k^2}}$$

<u>Idea of proof</u>: $\mathbb{P}(X_n - X_0 \ge t) = \mathbb{P}(e^{\lambda(X_n - X_0)} \ge e^{\lambda t}) \le$ (for any $\lambda > 0$; Markov)

$$e^{-\lambda t} \mathbb{E}[e^{\lambda(X_n - X_0)}].$$

Now $$\mathbb{E} e^{\lambda(X_n - X_0)} = \mathbb{E}[e^{\lambda(X_{n-1} - X_0)} \cdot \underbrace{\mathbb{E}[e^{\lambda(X_n - X_{n-1})} | \mathcal{F}_{n-1}]}_{\text{just need to bound this}}]$$

<u>Hoeffding's lemma</u>: Let $a < b$. For any random variable X with $a \le X \le b$ almost surely,

$$\mathbb{E}[e^{\lambda(X - \mathbb{E}X)}] \le e^{\lambda^2/8 (b-a)^2}$$

Back to showing $\mathbb{P}^0(A_\ell) = 1$.

<u>martingale</u>: Position − accumulated drift.

Fix $U \subsetneq \mathbb{Z}^d$, U conn., $0 \in U$. Let $\hat{X}_n$ be the random walk with trans. prob. $\hat{P}_U$, starting at 0 and stopped when ~~existing~~ exiting U.

Let $M_n = \hat{X}_n - \sum_{i=0}^{n-1} d(\hat{X}_i)$

where $d(x) = \sum_{e\in\{\pm e_i\}} e \cdot \hat{P}_\omega(x, x+e)$

(so that $d(x)\cdot \ell \ge \varepsilon_\ell$ by Kalikow's cond.)

M_n is a martingale (each coord is a martingale) with bdd. differences.

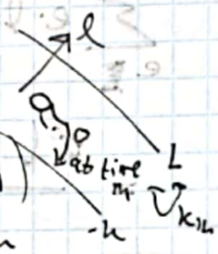
Hoeffding $\Rightarrow$ $\mathbb{P}(\|M_n\|_1 \ge \delta n) \le C e^{-c\delta^2 n}$

$(M_0=0)$.

$\Rightarrow \mathbb{P}(|M_n\cdot\ell| \ge \delta\cdot n) \le C\cdot e^{-c\delta^2 n}$

For $k,L>0$ let $U_{k,L} = \{z\in\mathbb{Z}^d : -k \le z\cdot\ell \le L\}$

What is the prob. to exit $U_{k,L}$ from the $-k$ side?



$$\hat{P}_{U_{k,L}}(X_{T_{U^c_{k,L}}}\cdot\ell < -k) \underset{\text{Assume } k<L}{=} \sum_{m\ge k}^{\infty} \hat{P}_{U_{k,L}}(X_{T_{U^c_{k,L}}}\cdot \ell < -k,\ T_{U^c_{k,L}} = m)$$

$$\overset{\text{Hoeffding}}{\le} \sum_{m=k}^{\infty} C\cdot e^{-c\,\varepsilon_\ell^2\cdot m} \le C(\varepsilon_\ell)\cdot e^{-c(\varepsilon_\ell)\cdot k}$$ on this event $M_m\cdot\ell \le -k - m\varepsilon_\ell$

By lemma 1: $\mathbb{P}^0(X_{T_{U^c_{k,L}}}\cdot \ell<-k) < C_1(\varepsilon_\ell)\, e^{-c(\varepsilon_\ell)k}$

taking k large and L to infinity,

$\mathbb{P}^0(\min_n X_n\cdot\ell \ge -k) > 0$

$\Rightarrow \mathbb{P}^0(O_\ell)<1$ $\quad O_\ell = \left\{\begin{array}{l}\limsup_{n\to\infty} X_n\cdot\ell = \infty \\ \liminf_{n\to\infty} X_n\cdot \ell=-\infty\end{array}\right\}$

Kalikow 0-1 law $\Rightarrow$ $\mathbb{P}^0(O_\ell)=0 \iff \mathbb{P}^0(A_\ell\cup A_{-\ell})=1$

Lastly, taking $k=L$, $\mathbb{P}^0(X_{T_{U^c_{L,L}}}\cdot\ell<-L) \le c(\varepsilon_\ell)\cdot e^{-c(\varepsilon_\ell)\cdot L}$

$\Rightarrow \mathbb{P}^0(\lim_{n\to\infty} X_n\cdot\ell = -\infty) = 0 \iff \mathbb{P}^0(A_\ell)=0.$

Scanned with CamScanner

To use lemma 1 we needed $\hat{\mathbb{P}}_{\mho_{k,L}}(\tau_{\mho^c_{k,L}} < \infty) = 1$

Indeed, $\{\tau_{\mho^c_{k,L}} = \infty\}$ entails that $|X_n \cdot \ell| \to \infty$.

Kalikow test:

Lemma: Let $\mathcal{F} = \{f : \{\pm e_i\} \to [0,1]$. f not identically zero$\}$

Then for all $\ell \in S^{d-1}$,

$$\varepsilon_\ell \geq \inf_{f \in \mathcal{F}} \frac{\mathbb{E}\left(\dfrac{\sum_e \omega(0,e) \cdot e \cdot \ell}{\sum_e \omega(0,e) \cdot f(e)} \right)}{\mathbb{E}\left(\dfrac{1}{\sum_e \omega(0,e) \cdot f(e)} \right)}$$

$\leftarrow$ only depends on $\omega(0,\cdot)$

Thus, if the RHS>0 then Kalikow's cond. holds.

Proof: Fix $\mho \subsetneq \mathbb{Z}^d$ conn., $0 \in \mho$ and $x \in \mho$

$$\sum_e e \cdot \ell \cdot \hat{P}_\mho(x, x+e) = \frac{\mathbb{E}^0\left(\sum_{n=0}^{\tau_{\mho^c}} \mathbb{1}_{X_n = x} \sum_e \omega(x, x+e) \cdot e \cdot \ell \right)}{\mathbb{E}^0\left(\sum_{n=0}^{\tau_{\mho^c}} \mathbb{1}_{X_n = x} \right)}$$

$$= \frac{\mathbb{E}^0\left(\sum_e \omega(x, x+e) \cdot e \cdot \ell \cdot E^0_\omega\left(\sum_{n=0}^{\tau_{\mho^c}} \mathbb{1}_{X_n = x} \right) \right)}{\mathbb{E}^0\left(E^0_\omega\left(\sum_{n=0}^{\tau_{\mho^c}} \mathbb{1}_{X_n = x} \right) \right)}$$

Examine: $E^0_\omega\left(\sum_{n=0}^{\tau_{\mho^c}} \mathbb{1}_{X_n = x} \right)$

Considering the first and later visits separately

$$E^0_\omega\left(\sum_{n=0}^{\tau_{\mho^c}} \mathbb{1}_{X_n = x} \right) = P^0_\omega(\tau_x < \tau_{\mho^c}) \, E^x_\omega\left(\sum_{n=0}^{\tau_{\mho^c}} \mathbb{1}_{X_n = x} \right)$$

τ_x: visit time of x

has a geom. dist. with success prob. $P^x_\omega(\tau_{\mho^c} < \tau_x)$

τ_x: return time to x

$$P^x_\omega(\tau_{\mho^c} < \tau_x) = \sum_e P^x_\omega(x, x+e) \cdot P^{x+e}_\omega(\tau_{\mho^c} < \tau_x)$$

$P^x_\omega(x,x+e) = \omega$

$$= P^0_\omega(\tau_x < \tau_{\mho^c}) \cdot \frac{1}{\sum_e \omega(x, x+e) \cdot P^{x+e}_\omega(\tau_{\mho^c} \leq \tau_x)}$$

Plugging back we get:

$$\sum_e e \cdot \ell \cdot \hat{\mathbb{P}}_{\upsilon}(x, x+e) = \frac{\mathbb{E}^0\left(\dfrac{\sum_e w(x,x+e)\cdot e \cdot \ell}{\sum_e w(x,x+e)\cdot \dfrac{\mathbb{P}_\omega^{x+e}(\tau_{\upsilon^c} < \tau_x)}{\mathbb{P}_\omega^0(\tau_x < \tau_{\upsilon^c})}} \right)}{\mathbb{E}^0\left(\dfrac{1}{\sum_e \omega(x,x+e)\cdot \dfrac{\mathbb{P}_\omega^{x+e}(\tau_{\upsilon^c} < \tau_x)}{\mathbb{P}_\omega^0(\tau_x < \tau_{\upsilon^c})}} \right)}$$

expectation over ω.

Environ. is IID.

This is a fch. only of $(\omega(y,\cdot))_{y \neq x}$

We write:

$$\mathbb{E}^0 = \mathbb{E}_{(\omega(y,\cdot))_{y\neq x}} \mathbb{E}_{\omega(x,\cdot)}$$

together

$$\geq \inf_{f \in \mathcal{F}} \mathbb{E}_{\omega(x,\cdot)} \cdot \frac{\mathbb{E}^0\left(\dfrac{\sum_e \omega(x,x+e) e \cdot \ell}{\sum_e \omega(x,x+e)\, f(e)} \right)}{\mathbb{E}^0\left(\dfrac{1}{\sum_e w(x,x+e)\, f(e)} \right)}$$

$\blacksquare$

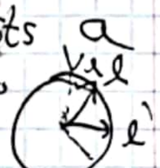
How sharp is Kalikow's cond.?
Does it characterize ballisticity in direction ℓ?
In dimension $d=1$, Kalikow's cond. is equivalent to ballisticity

Sznitman (2003) found examples of ballistic RWRE in $d \geq 2$ which do not satisfy Kalikow's cond.
(certain perturbations of simple random walk).

Sznitman's T cond. 2002:

Conditions $(T)_\gamma$, (T') and (T):

Let $\ell \in S^{d-1}$ and $\gamma \in [0,1]$.
Condition $(T)_\gamma | \ell$ is satisfied if there exists a neighborhood V_ℓ of ℓ s.t. $\forall \ell' \in V_\ell$ $\forall b > 0$



$$\limsup_{L\to\infty} \frac{1}{L^\gamma} \log \underbrace{\mathbb{P}^0 \left(H^{\ell'}_{-bL} < H^{\ell'}_{L} \right)}_{\leq e^{-cL^\gamma}} \leq 0,$$ In other words:
$\mathbb{P}^0(H^{-\ell}_{bL} < H^{\ell}_{L}) = e^{-\Omega(L^\gamma)}$ as $L \to \infty$

$$H_L^{\ell'} = \min\{n \geq 0 \mid X_n \cdot \ell \geq L\}$$

Condition $(T)|\ell$ is $(T)_\gamma|\ell$ for $\gamma = 1$.

Condition $(T')|\ell$ is $(T_\gamma)|\ell$ for every $\gamma \in (0,1)$

<u>Thm.</u> (Sznitman 02 and later extensions): In an IID, unif. elliptic environ in dimension $d \geq 2$, if either $(T)|\ell$, $(T')|\ell$ or $(T_\gamma)|\ell$ for $\gamma \in (0,1)$ holds then,

i) $\frac{X_n}{n} \to v \neq 0$ $\mathbb{P}^0$-a.s with $v \cdot \ell > 0$ (LLN)

ii) (annealed central limit thm):

$$\frac{1}{\sqrt{n}}(X_n - n \cdot v) \xrightarrow{d} N(0, \Sigma)$$

non-degenerate covariance matrix

(and also convergence to Brownian motion)

<u>Sharpness:</u> <u>$d=1$:</u> all these conditions are equivalent to transience.

<u>open:</u> In $d \geq 2$, are the cond. equivalent to ballisticity?

Also open: Transience $\Leftrightarrow$ ballisticity in IID unif. elliptic environment in dimension $d \geq 2$

Kalikow's cond. implies Sznitman's conditions, But not the other way (even in $d \geq 2$).

<u>Equivalence of cond.</u>

Berger - Drewitz - Ramirez 2012 introduced conition $(P^*)_M/\ell$ as follows: Let $M > 0$, $\ell \in S^{d-1}$. There exists a neighborhood V_ℓ of ℓ s.t $\forall \ell' \in V_\ell$:

$$\lim_{L \to \infty} L^M \mathbb{P}^0(H_{bL}^{-\ell'} < H_L^{\ell'}) = 0$$

In other words, $\mathbb{P}^0(H_{bL}^{-\ell'} < H_L^{\ell'}) = o(L^{-M})$

and proved that: $(P^*)_M|\ell$ for $M \geq 15d+5$ implies Sznitman's $(T')|\ell$ cond.

Guerra - Ramirez (2018): $(T')|\ell \Leftrightarrow (T)|\ell$.

Environment viewed from the point of view of the particle

Let X_n be a RWRE.
The environment process is $(\overline{\omega}_n)_{n\geq 0}^{\infty}$ where
$$\overline{\omega}_n = t_{X_n}\omega \quad \left(\begin{array}{l}\text{i.e. the translation of } \omega \text{ which} \\ \text{moves } X_n \text{ to the origin}\end{array}\right)$$
Here ω is the full environment, i.e. $(\omega(x,\cdot))_{x\in\mathbb{Z}^d}$
and $t_y\omega$ is the shifted environment $\omega(x-y,\cdot)_{x\in\mathbb{Z}^d}$

$(\overline{\omega}_n)_{n\geq 0}$ is a markov chain, even annealed.

We are interested in the limiting dist. of $\overline{\omega}_n$ as $n\to\infty$.

What would such a limit satisfy?

Any limiting dist. must be invariant to moves of the walk.

Markov chain on environments:

Given $\overline{\omega}_n$, the dist. of $\overline{\omega}_{n+1}$ is
$$\sum_{e\in\{\pm e_i\}} \overline{\omega}_n(0,e)\,\delta_{t_e\overline{\omega}_n}$$
Any limiting dist. must be invariant to these transitions.

It turns out that what is really useful is to have an invariant prob. distribution which is also absolutely cts with respect to the original environment measure P
← see Theorem 2.12 in Drewitz - Ramirez notes

Thm. (Kozlov 1985): Consider RWRE with env. measure P which is ergodic and elliptic.

Assume that ν is an invariant prob. measure on environ. which is abs. cont. w.r.t. P. Then the following are satisfied:

i) ν is equivalent to P. (i.e. P is abs. cont. w.r.t. ν)

ii) The environment process started with initial environ. from ν is stationary and ergodic.

iii) ν is the unique invariant prob measure which is abs. cont. w.r.t. P.

iv) The Cesaro averages $\frac{1}{n+1}\sum_{i=0}^{n}\delta_{\overline{w}_i}$ converge in dist. to ν when $\overline{w}_0$ is sampled from P

Remark: By compactness, the césaro averages always have subsequential limits.

One implication:

When an abs. cont. invariant measure ν exists then a law of large numbers is satisfied.

Moreover, $\frac{X_n}{n} \xrightarrow[n\to\infty]{} \int \sum_e \omega(0,e)\cdot e \cdot d\nu(\omega)$ P^0-a.s.

Moreover, if A_ℓ holds then $v\cdot \ell > 0$.

(directional transience implies ballisticity)